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ORIGINAL ARTICLE

Nonlinear isotropic elastic reduced Cosserat continuum as a possible model for geomedium and geomaterials. Spherical prestressed state in the semilinear material

Nonlinear reduced Cosserat continuum

Elena F. Grekova

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Abstract We suggest a nonlinear elastic reduced Cosserat continuum as a possible model for geomedium and geomaterials and also for a soil or rock with heterogeneities. If a medium has a block structure, or if it contains heterogeneities that may have their proper rotational dynamics, the presence of rotational degrees of freedom may influence wave propagation and stability of the medium. In the reduced Cosserat model, translations and rotations are independent, the medium resists to the rotation of each "particle" relatively to the background continuum, but it does not resist to the gradient of rotation. We consider a nonlinear spherical stress state in an isotropic elastic reduced Cosserat material. We write down small deviations from this nonlinear equilibrium. They coincide in form with the equations of the linear elastic isotropic reduced Cosserat continuum. Depending on the level of the stress and on the type of elastic energy, the equilibrium can be stable or unstable. In the domain of stability, shear–rotational wave demonstrates dispersion. There is a resonant frequency corresponding to the independent rotational oscillations. The bulk plane shear–rotational wave has a forbidden band of frequencies. In this zone, the shear–rotational wave localises near heterogeneities or external sources. We show that for a semilinear medium (medium with large deformations but linear dependence of stress on the strain tensor), strong compression leads to the material instability caused by shear perturbations, and strong tension for some class of parameters yields in instability caused by rotational perturbations.

Keywords Reduced Cosserat medium · Wave propagation · Instability · Localisation

Mathematics Subject Classifications (2010) 74B15 · 74B20 · 74L05 · 74L10 · 74J10 · 74M25

1 Introduction

There are many experimental evidences of the existence of rotational degrees of freedom in geodynamics on different scales: vortex structures, rotations of blocks of geomedium (Vikulin et al. 2011). In the state of predestruction in fault zones and near them, there exist rotations of rigid domains, coupled with localised shear bands, and also rotations of grains and aggregates characteristic for plastic and quasiplastic motion (Goldin 2004). It

E. F. Grekova (✉)
Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, 199178, Bolshoy pr. V.O., 61, St. Petersburg, Russia
e-mail: elgreco@pdmi.ras.ru

is observed in experiments on rock samples that in the process of plastic motion, a specific mesostructure is formed in the material, containing "rolling" rigid domains and soft bands where the shear is localised. A similar mesostructure is formed in seismic focal zones. Such a mesostructure can be formed due to the heterogeneous elastic properties of the geomedium, which has high contrasts in the seismic focal zone (Goldin 2005). In other words, rigid blocks and softer "bonds" between them exist when the material still reacts elastically, and in the process of predestruction, the bonds start to break.

We suggest a nonlinear elastic reduced Cosserat continuum as a possible model to describe the start of material shear–rotational instability phenomena and also consider waves in the domain of stability near a nonlinear prestressed state. We shall develop the model of this continuum basing upon fundamental principles of mechanics and then draw some qualitative conclusions that will allow to seismologists to judge if this model is useful for modelling instabilities and wave motion in the near field and if it is worth to develop it further.

A medium with a block or granular structure, in principle, can be also described in the frame of the classical media, whose particles only have translational degrees of freedom. However, in this case, we have to consider a very heterogeneous elastic structure, and this is a much more complex problem. If we do not need a detailed consideration of the processes inside hard particles and softer boundaries, then we can simplify the theory suggesting an effective medium, where a typical heterogeneity or block with surrounding medium would play a role of a point body of the continuous medium, able to move and rotate. Then the model of the medium becomes more complex, but the problem to consider is simpler. Also, when choosing a model for an effective homogeneous medium, we may neglect in some cases rotational degrees of freedom. In the domain of stability, for low frequencies, when the typical wavelength is much larger than a typical heterogeneity, the heterogeneous structure only changes the effective parameters of the medium. However, we cannot estimate in advance the wavelength by its frequency, since the Cosserat-type media are very dispersive. We shall see that the reduced Cosserat continuum has very low shear phase and group velocities near a certain frequency. For high frequencies in the stable equilibrium state, the medium particle has not enough time to perform noticeable rotation relatively to its neighbours, and the medium also behaves similar to the classical one, though some anomalies in wave velocities may be observed. In particular, in the domain of stability, the shear wave is faster at high frequencies than at low ones and for some media and stress states even may become faster than the P-wave. In this work, we shall concentrate our attention on cases when the internal structure of the medium does play a significant role, and we cannot neglect rotational degrees of freedom.

Reduced Cosserat medium is a continuum, whose particles can perform large rotations and translations. Rotations are kinematically independent on translations. Reduced Cosserat continuum does not react to the gradient of rotation but resists to the translational strain and to the rotation of each particle relatively to the background medium. The latter means that if we model a medium consisting of blocks, each block may rotate relatively to the "net" of centres of mass of neighbouring blocks. If we model a real material containing heterogeneities as an effective homogeneous medium, each heterogeneity may rotate relatively to the surrounding medium. However, we suppose that there is no interaction that decreases relative proper rotation of the neighbouring blocks or heterogeneities. Indeed, blocks may even tend to rotate in the opposite directions due to contact interaction. In block media, often used for modelling the geomedium (Vikulin et al. 2011) in rocks, soils and granular materials, there is no ordered structure of rotations, which is present in magnetic materials due to exchange interaction. Therefore, to model these media, it is better to exclude couples that resist to the relative rotation of neighbouring particles. Perhaps, it would be appropriate to consider some couples that favour such a relative rotation, but this will mean that the gradient of rotation can become infinitely large, and for the time being, there are no continuum theories that can deal with this. Thus, as a first step in the modelling, we limit ourselves to the case when nothing works on the gradient of the angular velocity (the medium does not resist to the relative rotation of its particles).

Such a medium in its linear isotropic elastic variant was suggested first by Schwartz et al. (1984) to model granular materials. We call it "reduced Cosserat model", since it is a particular (degenerative) case of the full variant of the Cosserat model, where the medium resists to the gradient of rotation. The full Cosserat continuum was described by the brothers E.&F. Cosserat (1909) more than a century ago. After that, this theory was forgotten for 50 years, and then C. Truesdell called the attention of the scientific society to this theory. It was further developed in many works, and the nonlinear case was considered in Kafadar and Eringen (1971). Many researchers, for instance, Vardoulakis and Unterreiner (1995) and Suiker et al. (2001), consider granular-type media in terms of full Cosserat continua, and there are various special cases of the full Cosserat continuum describing electromagnetic media (see Maugin 1988; Grekova and Zhilin 2001).

We will show that in the reduced Cosserat continuum, the stress tensor is, generally speaking, not symmetric, but the couple stress tensor is zero. We will see that when the rotational dynamic term is negligible (in statics or quasistatics in the rotational sense) and the external volume couples are zero, the stress is symmetric, like in the classical theory. However, it does not mean that we can neglect the rotational degrees of freedom even in this situation. We shall consider an isotropic material in the spherically symmetric stress state, quasistatically reached from a natural (stress free) state. When we write down the equations for the small deviations from the nonlinear equilibrium, we shall see that sufficiently large pressure or large tension for some materials can yield in instabilities. Under large compression, the material fails via shear low-frequency perturbations, and under large tension, the material may fail under rotational perturbations of a certain resonant frequency. In the stable domain, the shear–rotational wave has a strong dispersion near a certain zone of frequencies, where its propagation is forbidden. In this zone, shear–rotational waves will be localised near heterogeneities. We restrict ourselves to the elastic case without temperature and heat effects.

2 Basic equations of the nonlinear elastic reduced Cosserat continuum

2.1 Definitions and basic facts

We recall some definitions and facts from the continuum mechanics. In a nonlinear medium, we have to distinguish reference configuration (before deformation) and actual configuration (after deformation).

The *classical elastic medium* is a continuum consisting of point masses. Each point is characterized by the position vector $\mathbf{R}$ in the actual configuration ($\mathbf{r}$ in the reference configuration), and it has mass density ρ. The elastic energy U depends upon the strain $\overset{\circ}{\nabla}\mathbf{R}$, where $\overset{\circ}{\nabla}$ is the gradient operator in the reference configuration (Fig. 1). Let ∇ be the gradient operator in the actual configuration.

In the *full Cosserat continuum*, each particle of the medium is an infinitesimal rigid body, called point body in the terminology by C. Truesdell. It is characterized by two kinematic tensorial fields: a position vector ($\mathbf{R}$ in the actual configuration, $\mathbf{r}$ in the reference configuration) and a turn-tensor (or rotation tensor) ($\mathbf{P}$ in the actual configuration, in the reference configuration we consider it to be equal to the identity tensor $\mathbf{E}$). If we "embed" into each point body, a rigid triad of vectors $\mathbf{d}_k$

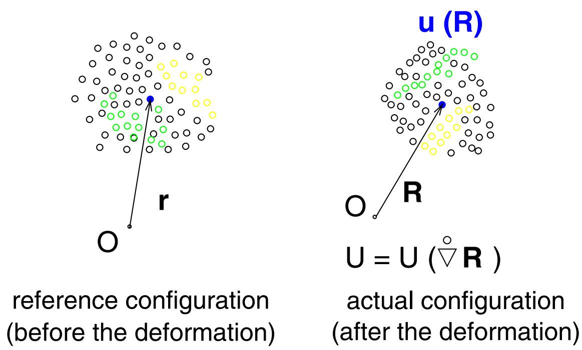

Fig. 1 Reference and actual configurations in the classical medium

characterizing its orientation, the rotated vector $\mathbf{d}_k$ equals $\mathbf{D}_k$, and the turn tensor $\mathbf{P}$ is such that $\mathbf{D}_k = \mathbf{P} \cdot \mathbf{d}_k$. In other words, $\mathbf{P} = \mathbf{D}_k \mathbf{d}^k$, where $\mathbf{d}^k$ is a dual basis ($\mathbf{d}^m \cdot \mathbf{d}_k = \delta_k^m$). Here and further, we omit the sign of the tensor product $\otimes$ and the sign of sum in the same sub and superscript (the Einstein summation rule).

The inertial characteristics of each point body are the mass density (ρ in the actual configuration, ρ_0 in the reference configuration) and the density of tensor of inertia, which we consider to be spherical for simplicity ($\rho I \mathbf{E}$ in the actual configuration, $\rho_0 I \mathbf{E}$ in the reference configuration). The strain energy in the elastic full Cosserat continuum depends on $\overset{\circ}{\nabla}\mathbf{R}, \mathbf{P}$, and on $\overset{\circ}{\nabla}\mathbf{P}$ (Fig. 2). If $\mathbf{t}_{(\mathbf{n})}$ is a force acting upon an infinitesimal surface with normal $\mathbf{n}$, there exist a (force) stress tensor $\boldsymbol{\tau}$ such that

$$\mathbf{t}_{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\tau}, \tag{1}$$

$\boldsymbol{\tau}$ is called the Cauchy stress tensor.

In the classical medium, as well as in the Cosserat continuum, the strain energy U cannot depend in an arbitrary way on its arguments. The constitutive equations have to satisfy the principle of material objectivity (the rigid motion of a piece of material must not influence the constitutive behaviour) formulated by Noll (1958).

In the classical medium, the law of balance of couples yields $\boldsymbol{\tau} = \boldsymbol{\tau}^\top$. In the Cosserat medium, the balance of couples has other terms and $\boldsymbol{\tau} \neq \boldsymbol{\tau}^\top$.

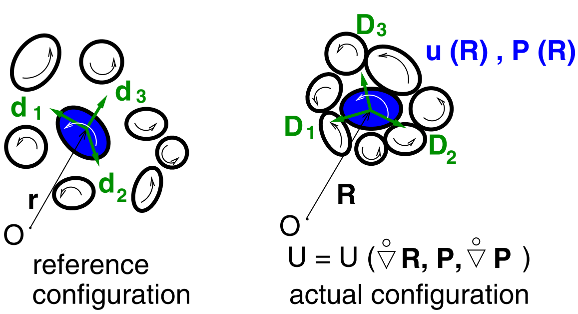

Fig. 2 Reference and actual configuration in the full Cosserat medium

If $\mathbf{m}_{(\mathbf{n})}$ is a couple acting upon an infinitesimal surface with normal $\mathbf{n}$, there exist a tensor $\boldsymbol{\mu}$ such that

$$\mathbf{m}_{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\mu}, \tag{2}$$

$\boldsymbol{\mu}$ is called the couple stress tensor.

The Cauchy stress tensor $\boldsymbol{\tau}$ works upon $\nabla \mathbf{v}$ (the gradient of translational velocity in the actual configuration) and upon $\boldsymbol{\omega}$ (angular velocity); $\boldsymbol{\mu}$ works upon $\nabla\boldsymbol{\omega}$ (gradient of angular velocity in the actual configuration). The balance of energy in the local form for the Cosserat medium is (see Kafadar and Eringen 1971)

$$\rho \dot{U} = \boldsymbol{\tau}^\top \cdot\cdot \nabla \mathbf{v} - \boldsymbol{\tau}_\times \cdot \boldsymbol{\omega} + \boldsymbol{\mu}^\top \cdot\cdot \nabla\boldsymbol{\omega}. \tag{3}$$

Here $\boldsymbol{\tau}_\times = \tau_{mn} \mathbf{i}_m \times \mathbf{i}_n$. To obtain this form of the balance of energy, we must use the laws of balance of force and torque.

Now let us introduce the *reduced Cosserat continuum* as such Cosserat continuum where nothing works on the gradient of angular velocity. We see then from Eq. 3 that this yields $\boldsymbol{\mu} = \mathbf{0}$. Thus, in the reduced Cosserat continuum, there are no couple stresses. However, the Cauchy stress tensor $\boldsymbol{\tau}$ is asymmetric, as in the full Cosserat continuum.

2.2 Constitutive equations

We obtain the constitutive equations for the nonlinear elastic reduced Cosserat continuum by a method suggested by P.A. Zhilin, which can be found in Grekova and Zhilin (2001). The law of balance of energy (Eq. 3) in the case of reduced Cosserat continuum is

$$\rho \dot{U} = \boldsymbol{\tau}^\top \cdot\cdot \nabla \mathbf{v} - \boldsymbol{\tau}_\times \cdot \boldsymbol{\omega} \tag{4}$$

and can be rewritten as

$$\rho \dot{U} = (\overset{\circ}{\nabla}\mathbf{R}^{-\top} \cdot \boldsymbol{\tau} \cdot \mathbf{P})^\top \cdot\cdot \dot{\mathbf{A}}, \tag{5}$$

where

$$\mathbf{A} = \overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{P} \tag{6}$$

is the Cosserat deformation tensor (see Grekova and Zhilin 2001). It is materially objective: Rigid translations of the material do not change it, and it is rotated together with the rigid rotations of the material. Any medium with the strain energy depending on $\mathbf{A}$ satisfies automatically the principle of material objectivity. Assuming $U = U(\mathbf{A})$, we obtain from Eq. 5 the constitutive equation

$$\boldsymbol{\tau} = \rho\, \mathring{\nabla}\mathbf{R}^\top \cdot \frac{\partial U}{\partial \mathbf{A}} \cdot \mathbf{P}^\top. \tag{7}$$

In the reduced Cosserat model, the strain energy does not depend on $\mathring{\nabla}\mathbf{P}$ (Fig. 3), contrary to the case of the full Cosserat continuum: Otherwise, in the law of balance of energy, there would appear terms, working on $\nabla\boldsymbol{\omega}$. On the level of modelling, it means that neighbouring point bodies may want to rotate in various directions and on average present resistance to the rotation of the point body under consideration (Fig. 4).

2.3 Dynamic laws

Since $\boldsymbol{\mu} = \mathbf{0}$, the dynamic laws (balances of linear and angular momenta) in the local form are:

$$\nabla \cdot \boldsymbol{\tau} + \rho\mathbf{K} = \rho\dot{\mathbf{v}}, \tag{8}$$

$$\boldsymbol{\tau}_\times + \rho\mathbf{L} = \rho I \dot{\boldsymbol{\omega}}, \tag{9}$$

where $\mathbf{K}$, $\mathbf{L}$ are external force and torque (couple) densities. This is a particular case (for zero couple stress) of the dynamic laws of the full Cosserat continuum, presented by Kafadar and Eringen (1971).

In the absence of volumetric torques in (quasi)statics, $\boldsymbol{\tau} = \boldsymbol{\tau}^\top$, as in the classical (non-polar) elasticity. But even in the case of quasistatic loading, it is important to take into account rotations for a proper analysis of stability. When arriving to the critical point, the material can change rapidly its structure, for instance, via shear–rotational motions.

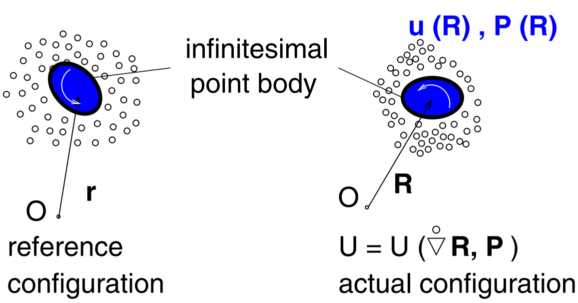

Fig. 3 Reference and actual configuration in the reduced Cosserat continuum

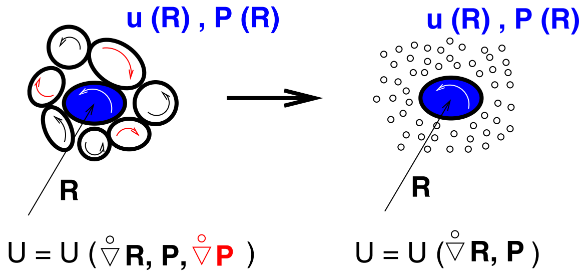

Fig. 4 Real material and reduced Cosserat medium: The neighbourhood of a particle resists to its rotation. There is no "positive elastic spring" resisting to the relative rotation of neighbouring particles

3 Spherically symmetrical stress state in an isotropic material

3.1 Equations for small deviations

Let us consider an isotropic homogeneous material subjected to the spherical deformation from the natural state ($\mathbf{A}_s = \mathring{\nabla}\mathbf{R}_s = A\mathbf{E}$, $\mathbf{P}_s = \mathbf{E}$). In the natural state $A = 1$, $A > 1$ corresponds to tension and $A < 1$ to compression. Here and further subindex $_s$ will refer to the value of a magnitude in this nonlinear state. Due to isotropy, the stress state is also spherical: $\boldsymbol{\tau}_s = \tau\mathbf{E}$.

Assume that $U(\mathbf{A})$ is approximable by Taylor series up to the second order near this nonlinear equilibrium. We consider small deviations from this state: $\mathbf{R} = \mathbf{R}_s + \mathbf{u}$, $\mathbf{P} = \mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}$. Here $\mathbf{u}$ is the translational displacement, and $\boldsymbol{\theta}$ is the vector of infinitesimal turn. Let us perturb the kinematic relation (6), constitutive equation (7) and dynamic laws (9) near the equilibrium and obtain the equations for deviations in general case.

After derivations (see the "Appendix"), we obtain the first law of dynamics by Euler (balance of forces):

$$\mathring{\nabla} \cdot \left(\rho_0 \left[\frac{\partial^2 U}{\partial \mathbf{A}^2} \right]_s \cdot\cdot \left(\mathring{\nabla}\mathbf{u} + A\boldsymbol{\theta} \times \mathbf{E} \right)^\top \right)$$
$$- \tau A^2 \mathring{\nabla} \times \boldsymbol{\theta} = \rho_0 \ddot{\mathbf{u}} \tag{10}$$

and the second law of dynamics by Euler (balance of couples):

$$-\tau_s A^2 \overset{\circ}{\nabla}\times\mathbf{u} + 2\tau_s A^3\boldsymbol{\theta} + \rho_0\left[A\left[\frac{\partial^2 U}{\partial \mathbf{A}^2}\right]_s \cdot\cdot \left(\overset{\circ}{\nabla}\mathbf{u} + A\boldsymbol{\theta}\times\mathbf{E}\right)^{\top}\right]_\times = \rho_0 I\ddot{\boldsymbol{\theta}}. \tag{11}$$

One may prove, using symmetry considerations, that the tensor of elastic constants in an isotropic material is equal to

$$\rho_0\left[\frac{\partial^2 U}{\partial \mathbf{A}^2}\right]_s = \lambda_s\mathbf{E}\mathbf{E} + 2\mu_s\left(\mathbf{i}_m\mathbf{i}_n\right)^S\left(\mathbf{i}^m\mathbf{i}^n\right)^S + 2\alpha_s\left(\mathbf{i}_m\mathbf{i}_n\right)^A\left(\mathbf{i}^n\mathbf{i}^m\right)^A, \tag{12}$$

where effective elastic constants λ_s, μ_s, α_s depend on the prestressed state. Indeed, an isotropic tensor of the fourth rank which does not change under permutation of the first and the second vector dyads, has this form.

3.2 Analogy with the linear elastic reduced Cosserat continuum

Define $\boldsymbol{\varphi} = A\,\boldsymbol{\theta}$. If we take into account Eq. 12, we see that the equations for perturbations (10) and (11) appear to be the same as those for the reduced linear Cosserat medium with zero initial stress, investigated by Grekova et al. (2009):

$$\left(\lambda' + 2\mu'\right)\overset{\circ}{\nabla}\overset{\circ}{\nabla}\cdot\mathbf{u} - \left(\mu' + \alpha'\right)\overset{\circ}{\nabla}\times\left(\overset{\circ}{\nabla}\times\mathbf{u}\right) + 2\alpha'\overset{\circ}{\nabla}\times\boldsymbol{\varphi} = \rho_0\ddot{\mathbf{u}}, \tag{13}$$

$$2\alpha'\overset{\circ}{\nabla}\times\mathbf{u} - 4\alpha'\boldsymbol{\varphi} = \left(\rho_0 I/A^2\right)\ddot{\boldsymbol{\varphi}}. \tag{14}$$

Here we have introduced effective elastic constants λ' and μ', analogous to the Lamé constants in the classical linear elastic medium, and effective constant of rotational resistance α':

$$\alpha' = \alpha_s - \frac{\tau A}{2}, \quad \mu' = \mu_s + \frac{\tau A}{2},$$
$$\lambda' + 2\mu' = \lambda_s + 2\mu_s. \tag{15}$$

Note that these equations coincide in form with the equations for the linear case. However, these equations are linearized equations for deviations near a given nonlinear prestressed state, and all the effective constants depend on it. Therefore, even in the domain of stability, the wave propagation is similar, but not identical to the linear case. If some of the effective elastic constants are negative, this will lead to instabilities that would be absent in the linear case.

Let us consider the case $\lambda_s + 2\mu_s > 0$, $\mu' > 0$, $\alpha' > 0$. Then Eq. 14 describe bulk wave propagation in the neighbourhood of the nonlinear spherical stress state. It inherits all the properties of the bulk wave propagation in the linear isotropic reduced Cosserat continuum that we will cite here (for the details, see Grekova et al. 2009):

– Depending on the effective constants of the medium, the shear wave at high frequencies may be faster (if $\lambda' + \mu' < \alpha'$) or slower (if $\lambda' + \mu' > \alpha'$) than the compression wave (Fig. 5).
– The compression wave is the same as in the classical medium.

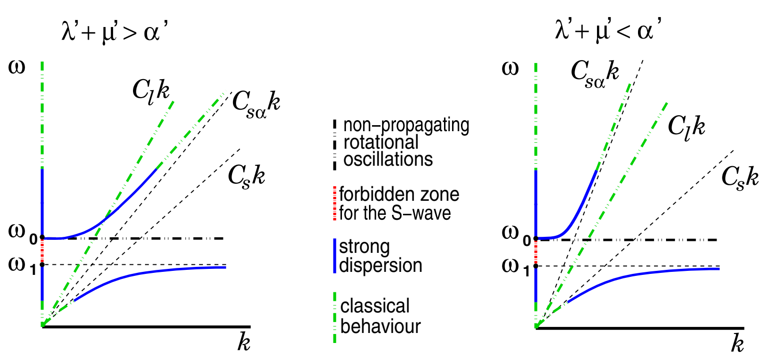

Fig. 5 Dispersion curves for the linear reduced isotropic elastic Cosserat continuum. $\omega_0^2 = 4\alpha' A^2/(\rho_0 I)$, $\omega_1^2 = \omega_0^2/(1+\alpha'/\mu')$, $C_s^2 = \mu'/\rho_0$, $C_{s\alpha}^2 = (\mu'+\alpha')/\rho_0$, $C_l^2 = (\lambda'+2\mu')/\rho_0$

- At low and high frequencies, the shear–rotational wave is almost non-dispersive; at high frequencies, it is faster than at low frequencies.
- There exists a special resonant frequency.

$$\omega_0 = \sqrt{4\alpha' A^2/(\rho_0 I)} = 2A\sqrt{(\alpha_s - \tau A/2)/(\rho_0 I)}, \quad (16)$$

corresponding to the independent rotational oscillations of point bodies.
- Shear–rotational wave has two branches. The lower branch has a horizontal asymptote

$$\omega_1 = \frac{\omega_0}{\sqrt{1+\alpha'/\mu'}} = 2A\sqrt{\frac{\alpha_s - \tau A/2}{\mu_s + \alpha_s}\cdot\frac{\mu_s + \tau A/2}{\rho_0 I}}. \quad (17)$$

- Shear–rotational wave does not propagate in the "forbidden zone" $[\omega_1;\omega_0)$.
- The compression plane wave has a transversal term, and the shear–rotational plane wave has a longitudinal term.
- Green functions for a harmonical point source $\mathbf{K} = \mathbf{K}_0 e^{i\omega t}$, $\mathbf{L} = \mathbf{L}_0 e^{i\omega t}$ are strongly frequency-dependent and if $\omega \in (\omega_1;\omega_0)$, the part of the wave is localised near the source. At $\omega = \omega_1$, if $\mathbf{L}_0 \neq \mathbf{0}$, we have a localised solution; at $\omega = \omega_0$, if $\mathbf{L}_0 \neq \mathbf{0}$, we have a resonant solution and a localised solution if $\mathbf{K}_0 \neq \mathbf{0}$, $\mathbf{L}_0 = \mathbf{0}$.

The dispersion curves for this medium are shown in Fig. 5 for two cases: $\lambda_s + \mu_s > \alpha_s$ (on the left) and $\lambda_s + \mu_s < \alpha_s$ (on the right). The dispersion relation for the shear–rotational wave is

$$k^2 = \frac{\omega^2}{C_S^2}\frac{1-\omega^2/\omega_0^2}{1-\omega^2/\omega_1^2} \overset{\text{def}}{=} \frac{\omega^2}{C_S^2} f^2(\omega). \quad (18)$$

Here $C_S^2 = \rho_0/(\mu_s + \tau A/2)$. One can show that at low frequencies

$$k \approx \frac{\omega}{C_S}\left(1 + \frac{\omega^2}{C_S^2}\frac{I}{8A^2}\right),$$

$$\omega \approx C_S k\left(1 - k^2\frac{I}{8A^2}\right), \quad (19)$$

and at high frequencies

$$k \approx \frac{\omega}{C_{s\alpha}}\left(1 - \frac{\omega_1^2}{\omega^2}\frac{\alpha'}{2\mu'}\right) = \frac{\omega}{C_{s\alpha}}\left(1 - \frac{1}{\omega^2}\frac{2A^2(\alpha_s - \tau A/2)^2}{\rho_0 I(\mu_s + \alpha_s)}\right), \quad (20)$$

$$\omega \approx C_{s\alpha}k\left(1 + k^{-2}\frac{2A^2}{I\left(1+\frac{\mu_s+\tau A/2}{\alpha_s-\tau A/2}\right)^2}\right), \quad (21)$$

where $C_{s\alpha}^2 = \rho_0/(\mu_s + \alpha_s)$. We see that at the limit of high and low frequencies, a weak dispersion is present.

If the material is semilinear (physically linear, geometrically nonlinear), i.e. λ_s, μ_s, α_s do not depend on the stress state, expressions (16) and (17) give that at high tension, when still $\alpha' > 0$, ω_1 and ω_2 come closer and tend to zero. However, this is not true for a general kind of strain energy.

Localisation at heterogeneities with weak contrasts in mass and inertia moment densities Let us consider infinitesimal heterogeneities in mass density and moment of inertia: $\rho_0 = \rho_{(0)} + \varepsilon\rho_{(1)}$, $\rho_0 I = I'_{(0)} + \varepsilon I'_{(1)}$, and let ω be a frequency of the incident plane wave. Suppose that the heterogeneity is localised at the origin of the coordinate system: $\rho'_1 = \rho_1\delta(\mathbf{r})$, $I'_1 = I_1\delta(\mathbf{r})$. We shall show that a part of the wave is localised near the heterogeneity if $\omega \in (\omega_1;\omega_0)$.

We look for the solution of Eq. 14 in the form of Born series, developing the solution in degrees of ε: $\mathbf{u} = \mathbf{u}_{(0)} + \varepsilon\mathbf{u}_{(1)} + \ldots$, $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{(0)} + \varepsilon\boldsymbol{\varphi}_{(1)} + \ldots$ We shall have in zero approximation a plane wave solution $\mathbf{u}_{(0)} = \mathbf{U}_{(0)}e^{(i\omega t + \mathbf{k}\cdot\mathbf{r})}$, $\boldsymbol{\varphi}_{(0)} = \boldsymbol{\Phi}_{(0)}e^{(i\omega t+\mathbf{k}\cdot\mathbf{r})}$. The next approximation gives us

$$\left(\lambda' + 2\mu'\right)\mathring{\nabla}\mathring{\nabla}\cdot\mathbf{u}_{(1)} - \left(\mu' + \alpha'\right)\mathring{\nabla}\times\left(\mathring{\nabla}\times\mathbf{u}_{(1)}\right) + 2\alpha'\mathring{\nabla}\times\boldsymbol{\varphi} + \rho_1\omega^2\mathbf{U}_{(0)}e^{i\omega t}\delta(\mathbf{r}) = \rho_0\ddot{\mathbf{u}}_{(1)}, \quad (22)$$

$$2\alpha'\mathring{\nabla}\times\mathbf{u}_{(1)} - 4\alpha'\boldsymbol{\varphi}_{(1)} + I'_1\omega^2\boldsymbol{\Phi}_{(0)}e^{i\omega t}\delta(\mathbf{r}) = (\rho_0 I/A^2)\ddot{\boldsymbol{\varphi}}_{(1)}. \quad (23)$$

The approximation of the first order can be found via expressions for the Green functions (Grekova et al. 2009), where $\rho_1\omega^2\mathbf{U}_{(0)}e^{i\omega t}$, $I'_1\omega^2\boldsymbol{\Phi}_{(0)}e^{i\omega t}$ play role of external harmonic force and couple, respectively. We will write explicitly this approximation, since there are typographic errors in Grekova et al. (2009). If $\omega \neq \omega_1$, $\omega \neq \omega_0$,

$$\mathbf{u}_{(1)} = \frac{I_1' \omega^2 \boldsymbol{\Phi}_0 e^{i\omega t}}{8\pi\alpha'\mu'} \cdot \left(1 - \frac{\omega^2}{\omega_1^2}\right)^{-1} \frac{\mathbf{E}\times\hat{\mathbf{r}}}{r^2}$$
$$\left(1 + i\frac{\omega f(\omega)}{C_s} r\right) e^{-i\frac{\omega f(\omega)}{C_s} r} - \frac{\rho_1 \omega^2 \mathbf{U}_0 e^{i\omega t}}{4\pi\mu'(\lambda' + 2\mu')}$$
$$\left(\frac{\omega^2}{C_l^2} - \frac{\omega^2 f^2(\omega)}{C_s^2}\right)^{-1} \left(1 - \frac{\omega^2}{\omega_1^2}\right)^{-1}$$
$$\cdot\Bigg\{ -e^{-i\frac{\omega}{C_l} r} \left((\lambda' + \mu' - \alpha') \left(1 - \frac{\omega^2}{\omega_0^2}\right) + \alpha' \right)$$
$$\left(\frac{1}{r}\frac{\omega^2}{C_l^2}\hat{\mathbf{r}}\hat{\mathbf{r}} + \left(1 + i\frac{\omega}{C_l} r\right) \frac{\mathbf{E} - 3\hat{\mathbf{r}}\hat{\mathbf{r}}}{r^3} \right) + e^{-i\frac{\omega f(\omega)}{C_s} r}$$
$$\left(\frac{1}{r}\mathbf{E} \left(1 - \frac{\omega^2}{\omega_0^2}\right)(\lambda' + 2\mu') \right.$$
$$\left(\frac{\omega^2}{C_l^2} - \frac{\omega^2 f^2(\omega)}{C_s^2}\right)$$
$$+ \left((\lambda' + \mu' - \alpha') \left(1 - \frac{\omega^2}{\omega_0^2}\right) + \alpha' \right)$$
$$\left(\frac{1}{r}\frac{\omega^2 f^2(\omega)}{C_s^2}\hat{\mathbf{r}}\hat{\mathbf{r}} + \left(1 + i\frac{\omega f(\omega)}{C_s} r\right) \right.$$
$$\left.\left.\frac{\mathbf{E} - 3\hat{\mathbf{r}}\hat{\mathbf{r}}}{r^3} \right) \right) \Bigg\}, \tag{24}$$

$$\boldsymbol{\varphi}_{(1)} = e^{i\left(\omega t - \frac{\omega f(\omega)}{C_s} r\right)} \left(1 - \frac{\omega^2}{\omega_1^2}\right)^{-1}$$
$$\Bigg\{ -\frac{\rho_1 \omega^2 \mathbf{U}_0}{8\pi\mu'} \cdot \frac{\mathbf{E}\times\hat{\mathbf{r}}}{r^2} \left(1 + i\frac{\omega f(\omega)}{C_s} r\right)$$
$$- \frac{I_1' \omega^2 \boldsymbol{\Phi}_0}{16\pi\alpha'^2\mu'} \left(1 - \frac{\omega^2}{\omega_0^2}\right)^{-1}$$
$$\cdot \left(\frac{1}{r}\mathbf{E} \left(1 - \frac{\omega^2}{\omega_0^2}\right) \left(\frac{\omega^2}{C_{s\alpha'}^2} - \frac{\omega^2 f^2(\omega)}{C_s^2}\right) \right.$$
$$(\mu' + \alpha') + \frac{1}{r}\hat{\mathbf{r}}\hat{\mathbf{r}}\frac{\omega^2 f^2(\omega)}{C_s^2}\alpha' + \frac{1}{r^3}$$
$$\left.\left(1 + i\frac{\omega f(\omega)}{C_s} r\right)(\mathbf{E} - 3\hat{\mathbf{r}}\hat{\mathbf{r}})\alpha' \right) \Bigg\}, \tag{25}$$

where $\hat{\mathbf{r}} = \mathbf{r}/|r|$. We see that in the forbidden zone for the shear–rotational plane wave $(\omega_1; \omega_0)$, where $f^2(\omega) < 0$, the part of wave is localised near the heterogeneity (it decays exponentially). At $\omega = \omega_0$, we have a resonant solution for $\boldsymbol{\varphi}_{(1)}$, if $I_1' \boldsymbol{\Phi}_{(0)} \neq \mathbf{0}$; this means that our expansion in this situation is not valid. If $I_1' \boldsymbol{\Phi}_{(0)} = \mathbf{0}$, we have a localised wave $\mathbf{u}_{(1)}$ decaying as $1/r^3$. At $\omega = \omega_1$, we have a strongly localised wave if $I_1' \boldsymbol{\Phi}_{(0)} \neq \mathbf{0}$ (for the expressions of Green functions (see Grekova et al. 2009).

We see that in the isotropic medium in the spherical prestressed state, the compression wave propagates as in the classical medium, but even weak heterogeneities will "trap" a part of shear–rotational wave and keep it localised near them. This could be a theoretical explanation of the phenomenon of localisation of shear–rotational part of seismic waves, observed near the edge of the basin (Ghayamghamian and Nouri 2010). In the classical linear elastic medium, such a localisation does not exist. The author does not know either any works on classical isotropic nonlinear homogeneous elastic materials that would give a localisation of a linearized shear–rotational wave near weak heterogeneity in an infinite medium and no such effect for the P-wave. This is also in agreement with the well-known fact that shear waves are more sensible to the heterogeneous structure of the medium. Such a localisation may also lead to rupture of bonds between particles and formation of a mesostructure characteristic for predestruction state, but to consider this process, we would have to develop further the model.

4 Instabilities

Perhaps the most interesting feature of the medium under consideration is that for an apparently "nicely looking" strain energy, the spherically symmetric stress state can be unstable. For instance, let us consider a strain energy which is positively defined quadratic form of $\mathbf{A}$. This means that $[\frac{\partial^2 U}{\partial \mathbf{A}^2}]$ is constant, $\alpha_s = \alpha > 0$, $\mu_s = \mu > 0$, $\lambda = \lambda_s$, $\lambda_s + 2\mu_s > 0$. In such a material, we have from Eqs. 7 and 15

$$\tau = (3\lambda + 2\mu)(A - 1)/A^2 = 3K\frac{A - 1}{A^2}, \tag{26}$$

$$\alpha' = \alpha - \frac{A-1}{2A}(3\lambda + 2\mu) = \alpha - \frac{3}{2}K\frac{A-1}{A}, \quad (27)$$

$$\mu' = \mu + \frac{A-1}{2A}(3\lambda + 2\mu) = \mu + \frac{3}{2}K\frac{A-1}{A}, \quad (28)$$

$$\lambda' + 2\mu' = \lambda + 2\mu. \quad (29)$$

Here $K = \lambda + 2\mu/3$ is the bulk modulus. If $A < 1 - 2\mu/(3K + 2\mu)$, we have $\mu' < 0$. When $\mu' = 0$, $\tau = -2\mu$. One can easily show that the value $\mu' = 0$ can be reached at small deformations, if K/μ is very large. On the contrary, if K/μ is infinitesimal, the stress state in which $\mu' = 0$ corresponds to infinite deformations and (if μ is large) infinite stresses. When it is reached, this means that the shear–rotational wave, corresponding to Eq. 14 at low frequencies, converts to an exponential solution (its velocity becomes imaginary, see Fig. 5). There is a local loss of the stability of the material in such a stress state. Qualitatively, this is in agreement with experiments on ordinary polycrystalline ice and polycrystalline tremolite, which were performed to explain the mechanisms of deep earthquakes (see Kirby 1987): "confining pressure does not suppress the (shear) instability and, under some conditions, raising confining pressure lowers the differential stress at which it occurs". However, in the work cited above, the experiments were performed for non-zero deviatoric stress, so they cannot be considered as a quantitative confirmation of the present theory.

If $\alpha < 3K/2$, we have $\alpha' < 0$ at $A > 1 + \alpha/(3K/2 - \alpha)$. This means that rotational–shear perturbations at frequency ω_0 for such a tension will lead to the instability of the material. When $\alpha' = 0$, $\tau = 2\alpha$. If K/α is large, this state is reached at small deformations, and if it is infinitesimal, the required tension is infinitely large.

Note that a classical nonlinear isotropic homogeneous elastic material whose strain energy is a positively defined quadratic form of the strain tensor *cannot* lose stability under isotropic tension or compression. If instead of modelling a granular or block (i.e. strongly heterogeneous) material by means of the reduced Cosserat theory we average elastic properties of a real heterogeneous medium with positive elastic constants, for instance, by methods similar to used in Bakhvalov and Panasenko (1989), using a long wave approximation to obtain an effective classical elastic homogeneous material, the effective classical material will be stable: The loss of the stability happens at the level of mesostructure, and the long wave approximation cannot capture it. Next steps in the method of averaging would give non-classical (gradient) theory, for instance, Cosserat-type theory.

An earthquake happens due to a local shear instability of the geomedium. This work shows that such an instability for a certain class of block media can appear in a zone of high isotropic compression (or tension) even if shear stresses are zero. When it happens, the material does not support any quasistatic shear perturbation. Let us see what happens with the "forbidden zone" for the shear waves when the material reaches an unstable state under compression.

When the compression is high enough to almost reach the instability, $-\tau A \to 2\mu_s$, $A \to A_{\text{cr}} < 1$, where A_{cr} is a certain critical value of the deformation. This means that

$$\omega_1^2 \approx \frac{4{A_{\text{cr}}}^2}{\rho_0 I}(\mu_s + \tau A_{\text{cr}}/2) \to 0, \quad (30)$$

$$\omega_0^2 \to \frac{4{A_{\text{cr}}}^2}{\rho_0 I}(\alpha_s + \mu_s). \quad (31)$$

This is true for all kinds of "non-exotic" strain energy, i.e. if $A < 1$ at compression. Thus, we see that when the medium reaches instability under high compression, it becomes in many senses similar to a liquid: It cannot resist slow shear deformation, and it "absorbs" shear waves of frequencies lower than $2A_{\text{cr}}\sqrt{(\alpha_s + \mu_s)/(\rho_0 I)}$. This is true for a zone under high isotropic compression approaching the instability. At other compression levels, the forbidden zone for the shear wave $[\omega_1; \omega_0)$ will be above zero.

Seismological observations tell us that near earthquake centres often there are zones absorbing shear waves of a certain band of frequencies (Kopnichev and Sokolova 2008). They explain it by the presence of fluid in these zones. On the other hand, another possible mechanism could be rotational dynamics of blocks trapping a part of the shear wave energy.

A semilinear material has a very specific type of elastic energy. One can show that instabilities of the spherically symmetric stress state at high tension or high compression take place also for a large class of other materials, but this question is beyond the scope of this paper.

Can we see these phenomena in experiment? Indeed, the classical diagram by Roscoe, used to describe the behaviour of granular materials (see Schwedes 1975) and unconsolidated geomaterials (see Jones and Addis 1986), predicts yield at a certain level of hydrostatic compression, as well as at a certain spherically symmetric tension. Note that this yield is a material failure via shear motions.

In experiments by Bridgman on materials under hydrostatic pressure (see Bell 1984), there was observed that material changes its structure if the pressure is sufficiently high. Sometimes the volume changes instantly when the pressure is constant, but often the volume of the material does not suffer any jump; however, the resistance of the material to the load changes. The last case obviously corresponds to the structural changes in the material via shear deformations (since the volume is constant). It may be caused by the fact that at these levels, the grain structure of a solid starts to play an important role. The model for the start of the geomedium breaking or start of a quasiplastic motion in seismic focal zones has to take into account many other factors, for instance, the presence of fluid, but the model discussed in this work could be considered as a first step.

5 Qualitative comparison with experimental observations

The nonlinear elastic reduced Cosserat continuum suggested here as a possible model for the geomedium can be considered as an effective medium for a heterogeneous material with high contrasts in elastic and inertial properties, consisting of blocks and softer bonds between them. Wave propagation in the reduced Cosserat medium would roughly take into account (in the effective way) scattering in the block medium with high contrasts for wavelengths of order of a typical block or larger ones, which is difficult to model theoretically on the microstructural level considering a classical heterogeneous medium. We see that shear waves in the model of the reduced Cosserat medium are dispersive. In a certain domain of frequencies, they have a slower group velocity than low-frequency and high-frequency shear waves (though the latter ones may be attenuated in real experiments due to viscosity, or perhaps just not measured). This may correspond to the formation of coda waves near the centre of the earthquake, where the contribution of shear waves is more important than one of the compression waves.

We are still far from the quantitative comparison of the theoretical results presented in this article with experimental seismological observations. To make this comparison, we have to consider at least non-isotropic prestressed state (for instance, combination of compressive and shear stresses corresponding to a symmetric stress tensor), heterogeneous in space (with maximal stresses near the centre of the earthquake). We foresee that the presence of shear stresses in the nonlinear equilibrium will couple compression and shear waves and the wave number for both of them will have real and imaginary part. The latter one (apparent attenuation) will be significant in a certain domain of frequencies. The shear wave will be more influenced by the presence of rotations than the compression wave. Probably the zone approaching to the state of instability will not let pass shear waves in a wide band of frequencies, especially at low frequencies, and when moving away from the zone of maximal stress, more shear waves of low frequencies will pass through the medium.

The results presented in this paper are particular, since the compression or tension is isotropic. This leads to the fact that the P-wave has no dispersion, and shear wave is frequency dependent (with characteristic frequencies depending on the stress state), but its wave number either is real, or only has an imaginary part (in the zone $[\omega_1; \omega_0)$). In this zone, the amplitude of the standing wave decreases with the distance from the source. Its apparent attenuation is frequency dependent, determined by $|f(\omega)|$, which tends to infinity at ω_1 and decreases to zero at ω_0, so higher-frequency shear perturbations in this frequency zone decrease more slowly in space than lower-frequency perturbations. This is in a qualitative agreement with experimental observations (Sato et al. 2002; Tsujiura 1978) that tell us that in spite of viscous effects, attenuation of coda waves decreases when the frequency increases (in the zone up to 30 Hz) and that this effect is more pronounced for heterogeneous media with high contrasts. The fact that in the reduced Cosserat material with a strain energy which satisfies the requirement of strong ellipticity (e.g. positively defined strain energy) we have instabilities at high hydrostatic compression or high isotropic tension is also in agreement with the idea that a possible mechanism of an earthquake is a local shear instability of the medium reached under very high compression and with the Roscoe diagram applicable for the description of geomaterials' behaviour.

6 Conclusions

We present here the basic equations of the reduced elastic nonlinear Cosserat continuum. We have considered a nonlinear spherically symmetric stress state for an isotropic material with a strain energy, approximable by Taylor series up to the second order with respect to Cosserat deformation tensor, and have written the equations for small deviations near this equilibrium. We have shown that the equations for small deviations in this case are similar to the equations for the linear reduced Cosserat continuum with initial zero stress. We have shown that in the domain of stability, we observe dispersion, polarization, localisation phenomena for shear–rotational waves at heterogeneities, and sources in the forbidden zone of frequencies, where this wave does not propagate in the homogeneous medium. The latter phenomena are reported in experimental works on seismic wave localisation.

The effective elastic constants may be negative at high tension or high compression in some materials. We considered in details the case of the semilinear material and have shown that under sufficiently large hydrostatic compression, it becomes unstable with respect to low-frequency shear perturbations. Many materials, in particular, soils, present this behaviour in experiments. If an additional condition for elastic constants is satisfied, at large tension the semilinear material fails via shear–rotational perturbations at a certain frequency.

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Appendix: Derivation of the equations for small deviations near a nonlinear spherical prestressed state in an isotropic homogeneous elastic reduced Cosserat medium

Let us introduce first Piola–Kirchhoff stress tensor:

$$\mathbf{T} \overset{\text{def}}{=} \overset{\circ}{\nabla}\mathbf{R}^{-\top} \cdot \boldsymbol{\tau} \, \det \overset{\circ}{\nabla}\mathbf{R} \tag{32}$$

and write down the equations of balance of force and torque (Eq. 9) in its terms:

$$\begin{aligned} &\overset{\circ}{\nabla} \cdot \mathbf{T} + \rho_0 \mathbf{K} = \rho_0 \dot{\mathbf{v}}, \\ &\left[\overset{\circ}{\nabla}\mathbf{R}^\top \cdot \mathbf{T} \right]_\times + \rho_0 \mathbf{L} = \rho_0 I \dot{\boldsymbol{\omega}}. \end{aligned} \tag{33}$$

Constitutive Eq. 7 looks as

$$\mathbf{T} = \rho_0 \frac{\partial U}{\partial \mathbf{A}} \cdot \mathbf{P}^\top. \tag{34}$$

We consider a nonlinear spherical prestressed state denoted by subindex $_s$: $\boldsymbol{\tau}_s = \tau \mathbf{E}$ (then $\mathbf{T}_s = T\mathbf{E}$, $T = \tau A^2$), $\overset{\circ}{\nabla}\mathbf{R}_s = A\mathbf{E}$, $\mathbf{P}_s = \mathbf{E}$ and small deviations from this state (subindex 1 will refer to the term of the first order): $\mathbf{R} = \mathbf{R}_s + \mathbf{u}$, $\mathbf{P} \approx \mathbf{E} + \boldsymbol{\theta} \times \mathbf{E}$, $\mathbf{T} \approx \mathbf{T}_s + [\mathbf{T}]_1$, $\mathbf{u} = o(1)$, $\boldsymbol{\theta} = o(1)$, $[\mathbf{T}]_1 = o(1)$. Note that $[\mathbf{P}]_1^\top = (\boldsymbol{\theta} \times \mathbf{E})^\top = -\boldsymbol{\theta} \times \mathbf{E}$, $[\boldsymbol{\omega}]_1 = \dot{\boldsymbol{\theta}}$ and $\mathbf{T}_s = \rho_0 \left[\frac{\partial U}{\partial \mathbf{A}}\right]_s$. We shall derive first-order approximations for the strain and stress tensors.

$$\begin{aligned} [\mathbf{A}]_1 &= \left[\overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{P} \right]_1 = \left[\overset{\circ}{\nabla}\mathbf{R} \right]_1 \cdot \mathbf{P}_s + \left[\overset{\circ}{\nabla}\mathbf{R} \right]_s \cdot [\mathbf{P}]_1 \\ &= \overset{\circ}{\nabla}\mathbf{u} + A\mathbf{E} \cdot (\boldsymbol{\theta} \times \mathbf{E}) = \overset{\circ}{\nabla}\mathbf{u} + A\boldsymbol{\theta} \times \mathbf{E}. \end{aligned} \tag{35}$$

Let us calculate $[\mathbf{T}]_1$ using Eq. 34:

$$\begin{aligned}[\mathbf{T}]_1 &= \rho_0 \left[\frac{\partial U}{\partial \mathbf{A}}\right]_1 \cdot \mathbf{P}_s{}^\top + \rho_0 \left[\frac{\partial U}{\partial \mathbf{A}}\right]_s \cdot [\mathbf{P}]_1^\top \\ &= \rho_0 \left[\frac{\partial U}{\partial \mathbf{A}}\right]_1 - \rho_0 \left[\frac{\partial U}{\partial \mathbf{A}}\right]_s \cdot (\boldsymbol{\theta} \times \mathbf{E}) \\ &= \rho_0 \left[\frac{\partial^2 U}{\partial \mathbf{A}^2}\right]_s \cdot\cdot [\mathbf{A}]_1^\top - \mathbf{T}_s \times \boldsymbol{\theta} \\ &= \rho_0 \left[\frac{\partial^2 U}{\partial \mathbf{A}^2}\right]_s \cdot\cdot \left(\overset{\circ}{\nabla}\mathbf{u} + A\boldsymbol{\theta} \times \mathbf{E}\right)^\top - T\boldsymbol{\theta} \times \mathbf{E}.\end{aligned} \tag{36}$$

The first-order approximation for Eq. 33 will look as

$$\begin{aligned}&\overset{\circ}{\nabla} \cdot [\mathbf{T}]_1 = \rho_0 \ddot{\mathbf{u}}, \\ &\left[\left[\overset{\circ}{\nabla}\mathbf{R}^\top \cdot \mathbf{T}\right]_\times\right]_1 = \rho_0 I \ddot{\boldsymbol{\theta}}.\end{aligned} \tag{37}$$

Let us proceed with the last equation, keeping in mind that $[\boldsymbol{\theta} \times \mathbf{E}]\times = -2\boldsymbol{\theta}$ and expression (36).

$$\begin{aligned}&\left[\left[\overset{\circ}{\nabla}\mathbf{R}^\top \cdot \mathbf{T}\right]_\times\right]_1 \\ &\quad = \left[\left[\overset{\circ}{\nabla}\mathbf{R}\right]_1^\top \cdot \mathbf{T}_s\right]_\times + \left[\overset{\circ}{\nabla}\mathbf{R}_s{}^\top \cdot [\mathbf{T}]_1\right]_\times \\ &\quad = T\left[\overset{\circ}{\nabla}\mathbf{u}^\top\right]_\times + [A[\mathbf{T}]_1]_\times \\ &\quad = -T\overset{\circ}{\nabla} \times \mathbf{u} \\ &\qquad + \left[\rho_0 A \left[\frac{\partial^2 U}{\partial \mathbf{A}^2}\right]_s \cdot\cdot \left(\overset{\circ}{\nabla}\mathbf{u} + A\boldsymbol{\theta} \times \mathbf{E}\right)^\top\right]_\times \\ &\qquad - [TA\boldsymbol{\theta} \times \mathbf{E}]_\times \\ &\quad = -T\overset{\circ}{\nabla} \times \mathbf{u} + 2TA\boldsymbol{\theta} \\ &\qquad + \left[\rho_0 A \left[\frac{\partial^2 U}{\partial \mathbf{A}^2}\right]_s \cdot\cdot \left(\overset{\circ}{\nabla}\mathbf{u} + A\boldsymbol{\theta} \times \mathbf{E}\right)^\top\right]_\times .\end{aligned} \tag{38}$$

Taking into account the last equation and Eq. 36, we can rewrite Eq. 37 as

$$\begin{aligned}&\overset{\circ}{\nabla} \cdot \left(\rho_0 \left[\frac{\partial^2 U}{\partial \mathbf{A}^2}\right]_s \cdot\cdot \left(\overset{\circ}{\nabla}\mathbf{u} + A\boldsymbol{\theta} \times \mathbf{E}\right)^\top\right) - \overset{\circ}{\nabla} \times (T\boldsymbol{\theta}) \\ &\quad = \rho_0 \ddot{\mathbf{u}}, \\ &- T\overset{\circ}{\nabla} \times \mathbf{u} + 2TA\boldsymbol{\theta} \\ &\quad + \left[\rho_0 A \left[\frac{\partial^2 U}{\partial \mathbf{A}^2}\right]_s \cdot\cdot \left(\overset{\circ}{\nabla}\mathbf{u} + A\boldsymbol{\theta} \times \mathbf{E}\right)^\top\right]_\times = \rho_0 I \ddot{\boldsymbol{\theta}}.\end{aligned} \tag{39}$$

These are equations of the first order for the deviations from the nonlinear spherical prestressed state. They are valid also for a heterogeneous material. If the material is homogeneous, using the fact $T = \tau A^2$, we obtain from them Eqs. 10 and 11.

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